Vector Analysis

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1 Describing arrows using 3-tuples (numerical arrays)

We may describe the geometric relationship of the Sun, Earth, and asteroid by drawing (big!) imaginary arrows between the three objects. Using a scale (like 1 inch = 1 AU, we may represent these big arrows using smaller arrows on our white board or paper.

While it possible to solve the Kepler problem using just these whiteboard arrows along with simple algebra, it is in fact a more arduous journey than making use of the analytic representation of arrows. In addition, if we use 3-tuples of real numbers to represent arrows, the whole scheme becomes admirably suited for computers. It is customary to choose basis arrows pointing in the directions of the x, y, and z axes. If we call these basis arrows $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Now comes the analytic scheme. Following tradition, we represent the basis arrows with 3-tuples thusly,

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1}$$

We then may write, for example,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \tag{2}$$

It is simple to define a dot product using $\mathbf{i} \cdot \mathbf{j} = 0$ etc. We get

$$\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \cdot \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} A_xB_x \\ A_yB_y \\ A_zB_z \end{pmatrix} \tag{3}$$

When programing, you may wish to think of this as

$$\mathbf{A}\cdot\mathbf{B} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \cdot \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} A_1B_1 \\ A_2B_2 \\ A_3B_3 \end{pmatrix} \tag{4}$$

Using $\mathbf{i}\times\mathbf{j} = \mathbf{k}$ etc we can implement the cross product of two vectors $\mathbf{F}$ and $\mathbf{S}$:

$$\mathbf{F}\times\mathbf{S} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \times \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} F_yS_z - F_zS_y \\ F_zS_x - F_xS_z \\ F_xS_y - F_yS_x \end{pmatrix} \tag{5}$$

When programming you may wish to write this as

$$\mathbf{F}\times\mathbf{S} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \times \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} F_2S_3 - F_3S_2 \\ F_3S_1 - F_1S_3 \\ F_1S_2 - F_2S_1 \end{pmatrix} \tag{6}$$

It is straightforward to verify that $\mathbf{F}\times\mathbf{S}\cdot\mathbf{F} = 0$.

2 Application to solving simultaneous linear equations

2.1 Two unknowns

Consider the simultaneous linear equations

$$u + 2v = -3 \tag{7}$$
$$2u - 2v = 6 \tag{8}$$

Using the array form of vectors, we may write this as

$$u\begin{pmatrix} 1 \\ 2 \end{pmatrix} + v\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} \tag{9}$$

This system of equations is solved graphically in figure 1.
We may use our vector analysis formalism to solve this system of equations analytically. Write the two equations as one vector equation

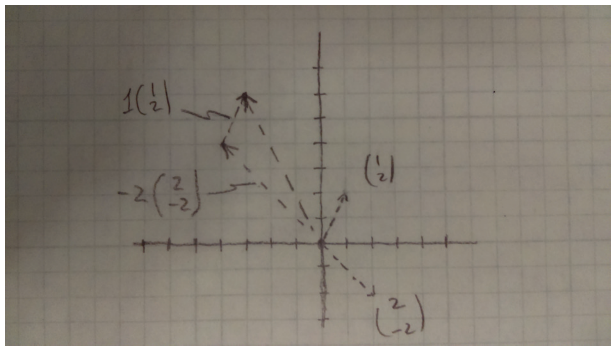

Figure 1: Geometric interpretation of simultaneous linear equations in two variables.

$$u(\mathbf{i}+2\mathbf{j}) + v(2\mathbf{i}-2\mathbf{j}) = -3\mathbf{i}+6\mathbf{j} \tag{10}$$

To get u, we eliminate v by cross multiplying the equation by $2\mathbf{i}-2\mathbf{j}$,

$$\begin{aligned} u(\mathbf{i}+2\mathbf{j})\times(2\mathbf{i}-2\mathbf{j}) + v(2\mathbf{i}-2\mathbf{j})\times(2\mathbf{i}-2\mathbf{j}) &= (-3\mathbf{i}+6\mathbf{j})\times(2\mathbf{i}-2\mathbf{j}) &(11)\\ u(-2-4)\mathbf{k} &= (6-12)\mathbf{k} &(12) \end{aligned}$$

from which is follows $u = 1$. Cross multiplying by $\mathbf{i}+2\mathbf{j}$ gets you v. Try it.

2.2 Three unknowns

We write this case in the index notation

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 &= B_1 &(13)\\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 &= B_2 &(14)\\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 &= B_3 &(15) \end{aligned}$$

This can be written in vectors as

$$x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + x_3\mathbf{A}_3 = \mathbf{B} \tag{16}$$

with

$$\mathbf{A}_1 = \begin{pmatrix} A_{11} \\ A_{21} \\ A_{31} \end{pmatrix} \quad \mathbf{A}_2 = \begin{pmatrix} A_{12} \\ A_{22} \\ A_{32} \end{pmatrix} \quad \mathbf{A}_3 = \begin{pmatrix} A_{13} \\ A_{23} \\ A_{33} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \tag{17}$$

To get x_1, first cross by $\mathbf{A}_2$

$$x_1\mathbf{A}_1 \times \mathbf{A}_2 + x_3\mathbf{A}_3 \times \mathbf{A}_2 = \mathbf{B} \times \mathbf{A}_2 \tag{18}$$

then dot by $\mathbf{A}_3$

$$x_1\mathbf{A}_1 \times \mathbf{A}_2 \cdot \mathbf{A}_3 + \mathbf{0} = \mathbf{B} \times \mathbf{A}_2 \cdot \mathbf{A}_3 \tag{19}$$

This can be solved for x_1. Using similar methods for x_2 and x_3 we have

$$x_1 = \frac{\mathbf{B} \times \mathbf{A}_2 \cdot \mathbf{A}_3}{\mathbf{A}_1 \times \mathbf{A}_2 \cdot \mathbf{A}_3} \quad x_2 = \frac{\mathbf{A}_1 \times \mathbf{B} \cdot \mathbf{A}_3}{\mathbf{A}_1 \times \mathbf{A}_2 \cdot \mathbf{A}_3} \quad x_3 = \frac{\mathbf{A}_1 \times \mathbf{A}_2 \cdot \mathbf{B}}{\mathbf{A}_1 \times \mathbf{A}_2 \cdot \mathbf{A}_3} \tag{20}$$